NEW SEMI–HAMILTONIAN HIERARCHY RELATED TO INTEGRABLE MAGNETIC FLOWS ON SURFACES

MISHA BIALY AND ANDREY MIRONOV

ABSTRACT. We consider magnetic geodesic flows on the 2-torus. We prove that the question of existence of polynomial in momenta first integrals on one energy level leads to a Semi-Hamiltonian system of quasi-linear equations, i.e. in the hyperbolic regions the system has Riemann invariants and can be written in conservation laws form.

1. INTRODUCTION

In this paper we introduce a new hierarchy of Semi-Hamiltonian system which is naturally related to integrable magnetic flows on surfaces.

More precisely we consider magnetic geodesic flows on two-torus. Consider one energy level and assume it admits a polynomial in momenta integral of motion. Then we prove that the system of quasi-linear equations on the coefficients is in fact Semi-Hamiltonian system. These systems were introduced by S.Tsarev and later on studied extensively. It is proved in [1] that these systems are integrable by the generalized hodograph method. A remarkable theorem by Sevennec [2] states that Semi-Hamiltonian property is equivalent to existence of two special coordinate systems in the space of field variables: Riemann invariants and Conservation laws. It is remarkable fact that both forms naturally appear for many systems. For example for Benney chains [3] and for geodesic flows [4] the Riemann invariants correspond to critical values of the integral, and Conservation laws are related to the invariant torii of the flow. In this paper the Semi-Hamiltonian property appears naturally in the same manner as we shall prove below.

The problem of existence of integrable systems in the presence of gyroscopic forces (which is equivalent to magnetic field) was studied by V.V.Kozlov and his students [5],[6],[7]. Topological obstructions to the integrability of mechanical systems on surfaces with magnetic fields were obtained in [7].

First of all let us recall some facts about geodesic flows on 2-torus (without magnetic field). If the geodesic flow is integrable then on the torus there are global semi-geodesic coordinates $ds^2 = g^2(t,x)dt^2 + dx^2$. This coordinates can be constructed using invariant Liouville torus having the diffeomorphic projection on the base $\mathbb{T}^2$. The existence of such a torus is proven in [8]. In the semi-geodesic coordinates the polynomial in momenta integral has the form

$$F_n = \sum_{k=0}^{n} a_k(t,x) \frac{p_1^{n-k}}{g^{n-k}} p_2^k,$$

where $a_{n-1} \equiv g$ *and* $a_n \equiv 1$. The coefficients satisfies the system

$$U_t + A(U)U_x = 0, \tag{1}$$

where $U = (a_0, \ldots, a_{n-2}, a_{n-1})^T$, and matrix A has the form

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \ldots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \ldots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \ldots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix}.$$

In [9] it is proven that in the case of integrals of degree 3 or 4 in the elliptic regions (where matrix A has all eigenvalues different and two of them are complex-conjugate) the integral can be reduced to integrals of the first or second degree or the metric is flat.

We proved in [4] that the system (1) is Semi-Hamiltonian. In this paper we generalize this result to the case of nontrivial magnetic field.

2. THE MAIN THEOREM

The geodesic flow on the torus $\mathbb{T}^2$ with the Riemannian metric $ds^2 = g_{ij}dx^i dx^j$ given by the Hamiltonian equations on $T^*\mathbb{T}^2$

$$\dot{x}^j = \frac{\partial H}{\partial p_j}, \ \dot{p}_j = -\frac{\partial H}{\partial x^j}, \ j = 1, 2,$$

where $H = \frac{1}{2}g^{ij}p_i p_j$. The function $F : T^*\mathbb{T}^2 \to \mathbb{R}$ is called the first integral of the geodesic flow if

$$\dot{F} = \{F, H\}_g = 0,$$

where $\{F, H\}_g$ is Poisson bracket

$$\{F, H\}_g = \sum_{j=1}^{2} \left(\frac{\partial F}{\partial x^j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial x^j} \frac{\partial F}{\partial p_j} \right).$$

In the case of magnetic geodesic flow the Poisson bracket gets the form

$$\{F, H\}_{mg} = \sum_{j=1}^{2} \left(\frac{\partial F}{\partial x^j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial x^j} \frac{\partial F}{\partial p_j} \right) + \Omega \left(\frac{\partial F}{\partial p_1} \frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2} \frac{\partial H}{\partial p_1} \right),$$

where Ω is the magnetic field. The magnetic geodesic flow given by the Hamiltonian equations with respect to magnetic geodesic bracket $\{.,.\}_{mg}$

$$\dot{x}^i = \{x^i, H\}_{mg}, \ \dot{p}_i = \{p_i, H\}_{mg}. \tag{2}$$

Let us choose conformal coordinates (x, y) on the torus. The metric becomes the form $ds^2 = \Lambda(x, y)(dx^2 + dy^2)$.

We shall consider the problem of existence of polynomial in momenta integral of motion F of degree N on one energy level $H = \frac{p_1^2 + p_2^2}{2\Lambda} = \frac{1}{2}$. We can parameterize the fibres of the energy level over $\mathbb{T}^2$ as follows. Write

$$p_1 = \sqrt{\Lambda} \cos \varphi, \ p_2 = \sqrt{\Lambda} \sin \varphi.$$

The equations (2) becomes of the form

$$\dot{x} = \frac{1}{\sqrt{\Lambda}} \cos \varphi, \ \dot{y} = \frac{1}{\sqrt{\Lambda}} \sin \varphi,$$

$$\dot{\varphi} = \frac{\Lambda_y}{2\Lambda\sqrt{\Lambda}} \cos \varphi - \frac{\Lambda_x}{2\Lambda\sqrt{\Lambda}} \sin \varphi - \Omega.$$

The integral F on the energy level has the form

$$F = \sum_{k=-N}^{k=N} a_k e^{ik\varphi}, \tag{3}$$

where $a_k = u_k(x, y) + i v_k(x, y), \ a_{-k} = \bar{a}_k$.

The condition $\dot{F} = 0$ is equivalent to

$$(F)_x \cos \varphi + (F)_y \sin \varphi + F_\varphi \left(\frac{\Lambda_y}{2\Lambda} \cos \varphi - \frac{\Lambda_x}{2\Lambda} \sin \varphi - \Omega\sqrt{\Lambda} \right) = 0.$$

We substitute (3) in the last equation and collect terms with respect to $e^{ik\varphi}$. We get

$$\frac{(a_{k-1})_x + (a_{k+1})_x}{2} + \frac{(a_{k-1})_y - (a_{k+1})_y}{2i} + \frac{\Lambda_y}{2\Lambda} \frac{i(k-1)a_{k-1} + i(k+1)a_{k+1}}{2} -$$

$$- \frac{\Lambda_x}{2\Lambda} \frac{i(k-1)a_{k-1} - i(k+1)a_{k+1}}{2i} - ik\Omega\sqrt{\Lambda} a_k = 0, \tag{4}$$

where $k = 0, \ldots, N + 1$ (we assume $a_s = 0$ at $s > N$).

Let k be $N + 1$ in (4):

$$(a_N)_x - N\frac{\Lambda_x}{2\Lambda} a_N + \frac{1}{i} \left((a_N)_y - N\frac{\Lambda_y}{2\Lambda} a_N \right) = 0.$$

Multiply the last identity by $\Lambda^{-N/2}$

$$(a_N \Lambda^{-N/2})_x - i(a_N \Lambda^{-N/2})_y = 0.$$

Thus $a_N \Lambda^{-N/2}$ is a holomorphic function, consequently $a_N = \Lambda^{N/2}(\alpha + i\beta)$ for some constants α, β. Taking appropriate rotation in the plane (x, y) and dividing F by appropriate constant we can assume that $\alpha = 1, \beta = 0$. Thus we have $a_N = \Lambda^{N/2}$.

Notice that for $k = 0$ the equation (4) does not contain magnetic field and has the form

$$\frac{(a_{-1})_x + (a_1)_x}{2} + \frac{(a_{-1})_y - (a_1)_y}{2i} + \frac{\Lambda_y}{2\Lambda}\frac{ia_1 - ia_{-1}}{2} + \frac{\Lambda_x}{2\Lambda}\frac{a_{-1} + a_1}{2} = 0. \quad (5)$$

Let us introduce the notation

$$Q_k = \frac{(a_{k-1})_x + (a_{k+1})_x}{2} + \frac{(a_{k-1})_y - (a_{k+1})_y}{2i} +$$

$$+ \frac{\Lambda_y}{2\Lambda}\frac{i(k-1)a_{k-1} + i(k+1)a_{k+1}}{2} - \frac{\Lambda_x}{2\Lambda}\frac{(k-1)a_{k-1} - (k+1)a_{k+1}}{2}.$$

From equation (4) for $k = N$ we can find the magnetic field:

$$\Omega = \frac{Q_N}{iN\sqrt{\Lambda}a_N}.$$

Let us substitute this expression of Ω into (4) for every $k = 1, \ldots, N-1$. We get

$$NQ_k\Lambda^{N/2} = kQ_N a_k. \quad (6)$$

The equations (5) and (6) form a system of quasi-linear equations on $2N$ unknown functions $\Lambda, u_0, \ldots, u_{N-1}, v_1, \ldots, v_{N-1}$. This is a quasi-linear system of the form

$$A(U)U_x + B(U)U_y = 0,$$

where $U = (\Lambda, u_0, \ldots, u_{N-1}, v_1, \ldots, v_{N-1})^\top$. We shall write in the last section this system explicitly for $N = 2$. Our main result is

Theorem 1. *For any N, the quasi-linear system (5),(6) on coefficients of the integral F is Semi-Hamiltonian system.*

3. Riemann invariants

Consider F as a function on the unite circle $S^1 \subset \mathbb{C}$. It is a remarkable fact that the critical values of

$$F = \Lambda^{N/2}z^N + a_{N-1}z^{N-1} + \cdots + \Lambda^{N/2}z^{-N}$$

are Riemann invariants of the system (5), (6). Indeed, let $x_1, \ldots, x_{2N}$ be the set of distinct critical points (hyperbolic region) $x_i \in S^1 \subset \mathbb{C}$. Introduce $r_k = F(x_k), k = 1, \ldots, 2N$. From the identity

$$\dot{F} = F_x\dot{x} + F_y\dot{y} + F_\varphi\dot{\varphi} = 0$$

it follows that r_k are Riemann invariants, because having $F_\varphi = 0$ we are left with $F_x cos\varphi + F_y sin\varphi = 0$ and thus the equation on r_k follows:

$$(r_k)_x + \lambda_k (r_k)_y = 0, \ \lambda_k = \tan \varphi_k, \ x_k = e^{i\varphi_k}.$$

Let us check that $r_1, ..., r_{2N}$ form a regular coordinate system, that is Riemann invariants are functionally independent. Write

$$zF' = N\Lambda^{N/2} z^N + (n-1)a_{n-1} z^{N-1} + \cdots + a_1 z - a_{-1} z^{-1} - \cdots - N\Lambda^{N/2} z^{-N}.$$

Notice that the critical points $x_1, \ldots, x_{2N}$ are roots of zF' and so by Vieta formula

$$x_1 \ldots x_{2N} = -1. \tag{7}$$

For convenience we denote field variables as

$$(\mu_1, \ldots, \mu_{2N}) = (\Lambda^{N/2}, a_{N-1}, \ldots, a_0, \ldots, a_{1-N}).$$

Then

$$\frac{\partial r_k}{\partial \mu_s} = \frac{\partial F}{\partial \mu_s}(x_k) + F'(x_k)\frac{\partial x_k}{\partial \mu_s} = \frac{\partial F}{\partial \mu_s}(x_k).$$

Using (7) we have

$$\det\left(\frac{\partial r_k}{\partial \mu_s}\right) = (-1)^N \det \begin{pmatrix} x_1^{2N} + 1 & x_1^{2N-1} & \ldots & x_1 \\ \ldots & \ldots & \ldots & \ldots \\ x_{2N}^{2N} + 1 & x_{2N}^{2N-1} & \ldots & x_{2N} \end{pmatrix}.$$

Splitting the first column and again using (7) we get

$$\det\left(\frac{\partial r_k}{\partial \mu_s}\right) = (-1)^N (x_1 \ldots x_{2N} - 1)W = (-1)^{N+1} 2W.$$

Where $W = \prod_{i>j}(x_i - x_j)$ is the Vandermonde determinant. So we have that $(\mu) \leftrightarrow (r)$ is a regular change of variable near every point A in the strictly hyperbolic region.

Remark. The field variables Λ, a_0 are real, a_s and $a_{-s}, s > 0$ are complex conjugate. Therefore computation of $\left(\frac{\partial r_k}{\partial \mu_s}\right)$ is equivalent to the computation of $\frac{\partial(r_1,\ldots,r_{2N})}{\partial(\Lambda^{N/2}, a_0, u_1, \ldots, u_{N-1}, v_1, \ldots, v_{N-1})}$.

4. CONSERVATION LAWS

The aim of this section is to prove that the system (5),(6) can be written in the form of conservation laws. This system has many explicit conservation laws. For example, the real part of (4) for $k = N$ multiplied by $\Lambda^{\frac{1-N}{2}}$ has the form of conservation low

$$\left(u_{N-1}\Lambda^{\frac{1-N}{2}}\right)_x + \left(v_{N-1}\Lambda^{\frac{1-N}{2}}\right)_y = 0.$$

Another series of conservation laws can be obtained in the following way. The identity (4) at $k = 0$ gives us a conservation law

$$(\sqrt{\Lambda}u_1)_x - (\sqrt{\Lambda}v_1)_y = 0.$$

Similarly we can get this conservation law for the powers of the integral. Namely F^m generates the conservations law

$$(\sqrt{\Lambda}u_1^{(m)})_x - (\sqrt{\Lambda}v_1^{(m)})_y = 0.$$

where $a_1^{(m)} = u_1^{(m)} + iv_1^{(m)}$ are corresponding Fourier coefficients of F^m. So, we have infinitely many explicit conservation lows. Remarkably they are in fact polynomial in the field variables. However we do not know if it is possible to get by this method functionally independent conservation lows. For this reason we proceed in a different way. We show that is possible to generate functionally independent conservation laws by invariant tori of the magnetic flow, in a similar way we did it in [4] for geodesic flows.

Imaginary part of (4) for $k = N$ multiplied by $\Lambda^{\frac{1-N}{2}}$ gives us

$$\Omega\Lambda = \frac{1}{2N}\left(v_{N-1}\Lambda^{(1-N)/2}\right)_x - \frac{1}{2N}\left(u_{N-1}\Lambda^{(1-N)/2}\right)_y. \qquad (8)$$

Let $\varphi = f(x,y)$ be a surface invariant under the flow. The invariance condition reads as follows:

$$\frac{f_x \cos f}{\sqrt{\Lambda}} + \frac{f_y \sin f}{\sqrt{\Lambda}} + \frac{\Lambda_x \sin f}{2\Lambda\sqrt{\Lambda}} - \frac{\Lambda_y \cos f}{2\Lambda\sqrt{\Lambda}} + \Omega = 0$$

or equivalently multiplying by Λ

$$(\sqrt{\Lambda}\sin f)_x - (\sqrt{\Lambda}\cos f)_y + \Omega\Lambda = 0. \qquad (9)$$

Substituting (8) into (9) we get a conservation law corresponding to invariant surface $\varphi = f(x,y)$:

$$\left(\sqrt{\Lambda}\sin f + \frac{1}{2N}v_{N-1}\Lambda^{(1-N)/2}\right)_x - \left(\sqrt{\Lambda}\cos f + \frac{1}{2N}u_{N-1}\Lambda^{(1-N)/2}\right)_y = 0.$$

Let us show now that by this method we can get, in the strictly hyperbolic region, $2N$ conservation laws with functionally independent G_k,

$$G_k = \mathrm{Im}[\sqrt{\Lambda}z_k + \frac{1}{2N}a_{N-1}\Lambda^{(1-N)/2}], \ z_k = e^{i\varphi_k}.$$

Let Γ be the domain of strict hyperbolicity of F (F' has $2N$ distinct roots on the unite circle).

Proposition 2. *Denote by $\hat{\Gamma}$ the open dense subset of Γ defined by the condition that $\pm i$ is not among the roots of F. Let $A^* = (\mu_1^*, \ldots, \mu_{2N}^*)$ be any point of $\hat{\Gamma}$. Then in a neighborhood of A^* there exist $2N$ functionally independent conservation laws.*

Corollary 1. *Quasi-linear system (5),(6) is Semi-Hamiltonian at any point of Γ.*

Proof of Corollary. By Sevennec theorem and the Proposition 1 the system is Semi-Hamiltonian at any point of $\hat{\Gamma}$. But it is dense, so the condition of Semi-Hamiltonicity extends to the whole Γ. Applying Sevennec theorem again we have that also for $A^* \in \Gamma\backslash\hat{\Gamma}$ there are $2N$ independent conservations laws, but we were not able to construct them explicitly. This proves the Corollary. $\square$

Proof of Proposition. Define a neighborhood of A^* with the help of Riemann invariants $r_1, \ldots, r_{2N}$. Recall r_k are local coordinates.

Let $r_k^*, k = 1, \ldots, 2N$ are Riemann invariants of A^*. That is x_k^* are critical points of $F^* = (\Lambda^*)^{N/2}z^N + a_{N-1}^* z^{N-1} + \cdots + (\Lambda^*)^{N/2}z^{-N}$. Notice that all critical points x_k^* are non-degenerate since we are in the strictly Hyperbolic region. We shall assume that odd and even values of the index k correspond to maxima and minima respectively. Let $\varepsilon > 0$ be a small number. Define a neighborhood of F^*

$$R_\varepsilon = \{F : |r_k - r_k^*| < \frac{\varepsilon}{10}, k = 1, \ldots, 2N\}.$$

Here ε should be chosen smaller then $\frac{1}{4}\min|r_k^* - r_{k+1}^*|$.

So for any polynomial from R_ε it has critical points x_k close to x_k^* on the unite circle and critical values close to r_k^*. Define now $z_k(\varepsilon, \mu)$ as solutions of the equations on the unite circle:

$$F(z_k(\varepsilon, \mu), \mu) = r_k + (-1)^k \varepsilon,$$

z_k lies in a neighborhood of x_k. Since we assume that $x_1, x_3, \ldots$ are points of maxima and $x_2, x_4, \ldots$ are points of minima then it follows that $r_k + (-1)^k \varepsilon$ are regular values for any polynomial F with coefficients taken from R_ε (F should be restricted to a neighborhoods of x_k^*). Important for us is that $z_k(\varepsilon, \mu)$ depend smoothly on ε and μ. When we shrink ε to zero then $z_k(\mu, \varepsilon) \to x_k$ by the construction. In the following we shall decrease ε, but keeping away from zero, in order to get the needed neighborhood R_ε.

Define the following functions on R_ε:

$$G_k = \text{Im}[\sqrt{\Lambda}z_k + \frac{1}{2N}a_{N-1}\Lambda^{(1-N)/2}] =$$

$$\frac{1}{2i}\text{Im}\left[\sqrt{\Lambda}\left(z_k - \frac{1}{z_k}\right) + \frac{1}{2N}\Lambda^{(1-N)/2}(a_{N-1} - a_{1-N})\right]$$

(where z_k depend on μs implicitly). We claim that one can choose $\varepsilon > 0$ sufficiently small in order to have

$$\det\frac{\partial(G_1, \ldots, G_{2N})}{\partial(\mu_1, \ldots, \mu_{2N})}(A^*) \neq 0.$$

This would imply the claim. We have

$$\frac{\partial G_k}{\partial \mu_l}|_{\mu=\mu^*} = -\frac{\sqrt{\Lambda^*}}{2i}\left(1 + \frac{1}{(z_k^*)^2}\right)\frac{\frac{\partial F}{\partial \mu_l}|_{\mu=\mu^*}}{F'(\mu^*, z_k)} + R_{kl}^*.$$

Where R_{kl}^* contains all terms of explicit derivation of $\frac{\partial G_k}{\partial \mu_l}$.

We have

$$\det\left(\frac{\partial G}{\partial \mu}\right)|_{\mu=\mu^*} = \left(\frac{\sqrt{\Lambda^*}}{2i}\right)^{2N} \prod_{k=1}^{2N}\left(1 + \frac{1}{(z_k^*)^2}\right) \prod_{k=1}^{2N} \frac{1}{F'(\mu^*, z_k)} \times$$

$$\times \det\left[\left(\frac{\partial F(z_k^*, \mu)}{\partial \mu_l}\right)|_{\mu=\mu^*} + R_{kl}^*\left(\frac{1}{1 + \frac{1}{(z_k^*)^2}}\right) F'(\mu^*, z_k^*)\right].$$

Mention that when $\varepsilon \to 0$, $\frac{1}{1 + \frac{1}{(z_k^*)^2}} \to \frac{1}{1 + \frac{1}{(x_k)^2}}$ which is finite since $x_k \neq i$ by assumptions. Also $z_k^*(\varepsilon) \to x_k^*$ and so $F'(\mu^*, z_k^*) \to 0$. Therefore the determinant in brackets when $\varepsilon \to 0$ tends to the determinant of the matrix

$$\left(\frac{\partial F(x_k^*, \mu)}{\partial \mu_l}\right).$$

This is exactly the determinant of the matrix considered in the section 3. So it is equal $-2W(x_1^*, \ldots, x_{2N}^*)$ and does not vanish. But then it follows that for small $\varepsilon > 0$ $\det\left(\frac{\partial G}{\partial \mu}(A)\right) \neq 0$ in a neighborhood of A^*. Proposition 2 and Theorem 1 are proved.

5. Discussion

In this section we discus some open problems.

In the case of $n = 2$ the equations (5),(6) on functions Λ, u_0, u_1, v_1 have the form

$$(\sqrt{\Lambda} u_1)_x - (\sqrt{\Lambda} v_1)_y = 0,$$

$$\left(\frac{u_1}{\sqrt{\Lambda}}\right)_x + \left(\frac{v_1}{\sqrt{\Lambda}}\right)_y = 0,$$

$$(u_0)_x + 2\Lambda_x - \frac{v_1}{2\sqrt{\Lambda}}\left(\left(\frac{u_1}{\sqrt{\Lambda}}\right)_y - \left(\frac{v_1}{\sqrt{\Lambda}}\right)_x\right) = 0,$$

$$-(u_0)_y + 2\Lambda_y - \frac{u_1}{2\sqrt{\Lambda}}\left(\left(\frac{u_1}{\sqrt{\Lambda}}\right)_y - \left(\frac{v_1}{\sqrt{\Lambda}}\right)_x\right) = 0.$$

Introduce the new functions f, g :

$$f = \frac{u_1}{\sqrt{\Lambda}}, \; g = \frac{v_1}{\sqrt{\Lambda}}.$$

We have

$$f_x + g_y = 0,$$

$$(f\Lambda)_x - (g\Lambda)_y = 0,$$

$$(u_0)_x + 2\Lambda_x - \frac{1}{2}g(f_y - g_x) = 0,$$

$$-(u_0)_y + 2\Lambda_y + \frac{1}{2}f(f_y - g_x) = 0.$$

This system can be written in the form

$$A(U)U_x + B(U)U_y = 0,$$

where $U = (\Lambda, u_0, f, g)$,

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{1}{2}g \\ 0 & 0 & 0 & -\frac{1}{2}f \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{1}{2}g & 0 \\ 2 & -1 & \frac{1}{2}f & 0 \end{pmatrix}.$$

The problem of existence of periodic solution is very interesting. The systems of such form (non-evolution form) were considered in [10] from the point of view of blow-up analysis along characteristic curves. It would be very interesting to apply these ideas to our system.

Notice

$$g f_y = (fg)_y - f g_y = (fg)_y + f f_x = 0,$$
$$f g_x = (fg)_x - g f_x = (fg)_x + g g_y.$$

So we have

$$(u_0)_x + 2\Lambda_x + \frac{1}{2}(gg_x - ff_x) - \frac{1}{2}(fg)_y = 0,$$

$$-(u_0)_y + 2\Lambda_y + \frac{1}{2}(ff_y - gg_y) - \frac{1}{2}(fg)_x = 0.$$

Thus we have explicit conservation laws form for our system

$$f_x + g_y = 0,$$

$$(f\Lambda)_x - (g\Lambda)_y = 0,$$

$$(u_0 + 2\Lambda + \frac{1}{4}(g^2 - f^2))_x - (\frac{1}{2}fg)_y = 0, \tag{10}$$

$$(-u_0 + 2\Lambda - \frac{1}{4}(g^2 - f^2))_y - (\frac{1}{2}fg)_x = 0. \tag{11}$$

Lust two conservation lows are very interesting. Let us recall that a hyperbolic diagonal system

$$(r_i)_x + \lambda_i(r_1, \ldots, r_n)(r_i)_y = 0, i = 1, .., n$$

is Semi-Hamiltonian if

$$\partial_j \Gamma^k_{ki} = \partial_i \Gamma^k_{kj}, \ i \neq j \neq k,$$

where

$$\Gamma^k_{ki} = \frac{\partial_i \lambda_k}{\lambda_i - \lambda_k}.$$

It can be proved that a diagonal system is Semi-Hamiltonian if and only if it can be written in some coordinates as a system of conservation laws (see [2]). We have the diagonal metric (see [1]) $g_{ii} = H_i^2$, and Lame coefficients can be found from the over-determined system

$$\partial_k \ln H_i = \Gamma^i_{ik}.$$

By Pavlov–Tsarev theorem [11] if the Semi-Hamiltonian system has two conservation lows of the form

$$F_x + G_y = 0, \ F_y + H_x = 0,$$

then the corresponding metric g_{ii} is Egorov metric i.e.

$$\beta_{ij} = \beta_{ji}, \ \beta_{ij} = \frac{\partial_i H_j}{H_i}, \ i \neq j.$$

In such a case the metric is potential: $g_{ii} = \partial_i a(r)$ for a function a. So it follows that our system for $N = 2$ is Egorov Semi-Hamiltonian system, since we have two conservation lows (10),(11). By similar calculations one can check that for $N = 3$ our system is also Egorov system. It would be interesting to prove this fact for arbitrary N.

Another interesting problem is to find Poisson bracket of hydrodynamical type for the system in the form of Dubrovin–Novikov [12] or Ferapontov–Mokhov [13].

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M. BIALY, SCHOOL OF MATHEMATICAL SCIENCES, RAYMOND AND BEVERLY SACKLER FACULTY OF EXACT SCIENCES, TEL AVIV UNIVERSITY, ISRAEL
E-mail address: bialy@post.tau.ac.il

A.E. MIRONOV, SOBOLEV INSTITUTE OF MATHEMATICS AND LABORATORY OF GEOMETRIC METHODS IN MATHEMATICAL PHYSICS, MOSCOW STATE UNIVERSITY
E-mail address: mironov@math.nsc.ru